

THE FUBINI PRODUCT AND ITS APPLICATIONS

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ABSTRACT. The Fubini product of operator spaces provide a powerful tool for analysing properties of tensor products. In this paper we review the theory of Fubini products and apply it to the problem of computing invariant parts of dynamical systems. In particular, we study the invariant translation approximation property of discrete groups.

1. INTRODUCTION

The Fubini product of C^* -algebras was first defined and studied by J. Tomiyama [19, 20, 21, 22] and Wassermann [23]. It is now a standard tool in the study of operator algebras and operator spaces. See for instance [10, 1, 13, 11, 15, 14, 3]. Unfortunately, to our knowledge, a comprehensive treatment of the subject is missing from the literature.

In this paper, we try to remedy the situation. In Section 2, we give a survey of the theory of Fubini products and its applications. Most of the results in Section 2 appear scattered in numerous articles, most notably [20, 23, 9, 10, 15, 6, 12]. After briefly recalling some definitions concerning operator spaces in Subsection 2.1, we define the Fubini product and prove its fundamental properties regarding functoriality, intersections, kernels, relative commutants, invariant elements, combinations in Subsections 2.2-2.8. In Subsection 2.9, we review the relation between the operator approximation property and the slice map property.

In Section 3, we apply the results in Section 2 to the study of groups with the invariant translation approximation property (ITAP) of J. Roe [17, Section 11.5.3]. In Subsection 3.1, we recall the definition of the uniform Roe algebra. In Subsection 3.2, we analyse the uniform Roe algebra of a product space. Finally, in Subsection 3.3, we study the ITAP of product groups. We show that for countable discrete groups G and H , if G has the approximation property (AP) of Haagerup-Kraus [6], the product $G \times H$ has the ITAP if and only if H has the ITAP.

Finally, in Section 4, we study the crossed product version of the Fubini product.

2. THE FUBINI PRODUCT

2.1. Operator Spaces. For the sake of completeness, we start with some definitions concerning operator spaces. See [4, 16] for a complete treatment.

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Let \mathcal{H} be a Hilbert space and let $B(\mathcal{H})$ denote the Banach space of bounded linear operators on \mathcal{H} . The ideal of compact operators on \mathcal{H} is denoted $K(\mathcal{H}) \subseteq B(\mathcal{H})$.

Definition 2.1. An *operator space* on \mathcal{H} is a closed subspace of $B(\mathcal{H})$.

Definition 2.2. For $a \in B(\mathcal{H})$ and $b \in B(\mathcal{K})$, we write $a \otimes b$ for the corresponding element in $B(\mathcal{H} \otimes \mathcal{K})$. For subsets $A \subseteq B(\mathcal{H})$ and $B \subseteq B(\mathcal{K})$, we write $A \times B$ for the subset $\{a \otimes b \mid a \in A, b \in B\} \subseteq B(\mathcal{H} \otimes \mathcal{K})$ and $A \odot B$ for the linear span of $A \times B$.

Definition 2.3. Let $A \subseteq B(\mathcal{H})$ and $B \subseteq B(\mathcal{K})$ be operator spaces. We define the *tensor product* $A \otimes B$ as the norm closure of $A \odot B$ or, equivalently, the closed linear span of $A \times B$ in $B(\mathcal{H} \otimes \mathcal{K})$.

Remark 2.4. Suppose $A \subseteq B(\mathcal{H}_1)$, $A \subseteq B(\mathcal{H}_2)$, $B \subseteq B(\mathcal{K}_1)$ and $B \subseteq B(\mathcal{K}_2)$. Then for any $x \in A \odot B$, we have

$$\|x\|_{B(\mathcal{H}_1 \otimes \mathcal{K}_1)} = \|x\|_{B(\mathcal{H}_2 \otimes \mathcal{K}_2)}.$$

Lemma 2.5 ([19, Lemma 3]). Suppose $S, A \subseteq B(\mathcal{H})$ and $T, B \subseteq B(\mathcal{K})$. If $S \otimes T \subseteq A \otimes B$, then $S \subseteq A$ and $T \subseteq B$. \square

Definition 2.6. A linear map $\phi: A \rightarrow B$ of operator spaces is *completely bounded* if the map $\phi \odot \text{id}_D: A \odot D \rightarrow B \odot D$ is bounded for all operator spaces D . We write $\phi \otimes \text{id}_D$ for the extension $A \otimes D \rightarrow B \otimes D$.

Completely contractive and completely isometric maps are defined similarly.

Definition 2.7. The *completely bounded norm* of a completely bounded map $\phi: A \rightarrow B$ is

$$\|\phi\|_{\text{cb}} := \|\phi \otimes \text{id}_{K(\mathcal{H})}: A \otimes K(\mathcal{H}) \rightarrow B \otimes K(\mathcal{H})\|,$$

where \mathcal{H} is a separable infinite dimensional Hilbert space.

Lemma 2.8. Let $\phi: A \rightarrow B$ be a completely bounded map. Then we have

$$\|\phi\|_{\text{cb}} = \sup_n \|\phi \otimes \text{id}_{M_n}: A \otimes M_n \rightarrow B \otimes M_n\|.$$

Moreover, for any operator space D , we have

$$\|\phi \otimes \text{id}_D\| \leq \|\phi\|_{\text{cb}}.$$

\square

The following extension theorem of Wittstock says that $B(\mathcal{H})$ is an injective object in the category of operator spaces and completely contractive maps.

Theorem 2.9 ((Wittstock)). Let $S \subseteq A$ be operator spaces and let $\phi: S \rightarrow B(\mathcal{H})$ be a completely bounded map. Then there exists a completely bounded extension $\psi: A \rightarrow B(\mathcal{H})$ with $\|\psi\|_{\text{cb}} = \|\phi\|_{\text{cb}}$. \square

2.2. The Fubini Product.

Definition 2.10 ((cf. [20], [23])). Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. The *Fubini product* $F(S, T, A \otimes B)$ of S and T is defined as the set of all $x \in A \otimes B$ such that $(\phi \otimes \text{id}_B)(x) \in T$ for all $\phi \in A^*$ and $(\text{id}_A \otimes \psi)(x) \in S$ for all $\psi \in B^*$.

Remark 2.11. We have

$$S \otimes T \subseteq F(S, T, A \otimes B) \subseteq A \otimes B.$$

Lemma 2.12. *Let K, L be subspaces of A^*, B^* respectively such that the closed unit ball of K and L are weak*-dense in the unit balls of A^* and B^* respectively. Then for any $S \subseteq A, T \subseteq B$ operator spaces the Fubini product $F(S, T, A \otimes B)$ equals the set*

$$\{x \in A \otimes B \mid (\phi \otimes \text{id}_B)(x) \in T \text{ and } (\text{id}_A \otimes \psi)(x) \in S \text{ for all } \phi \in K, \psi \in L\}.$$

The assumptions are for instance satisfied if K and L are the set of normal linear functionals in faithful representations of A and B .

Proof. Suppose that $(\phi \otimes \text{id}_B)(x) \in T$ for all $\phi \in L$. We need to show that $(\phi \otimes \text{id}_B)(x) \in T$ for all $\phi \in A^*$. Let $\phi \in A^*$ and let $(\phi_n) \subseteq L$ be a bounded sequence (or net) converging pointwise (i.e. in the weak*-sense) to ϕ . Then $(\phi_n \otimes \text{id}_B)(z) \rightarrow (\phi \otimes \text{id}_B)(z)$ for all $z \in A \odot B$ and using norm boundedness of $\phi_n \otimes \text{id}_B$, an $\varepsilon/3$ -argument shows that the same holds for $z \in A \otimes B$. Since $(\phi_n \otimes \text{id}_B)(x) \in T$ by assumption we conclude that $(\phi \otimes \text{id}_B)(x) \in T$. \square

Definition 2.13. We say that $(S, T, A \otimes B)$ has the *slice map property* if

$$F(S, T, A \otimes B) = S \otimes T.$$

2.3. Functoriality.

Lemma 2.14. *For $i = 1, 2$, let $S_i \subseteq A_i$ and $T_i \subseteq B_i$ be operator spaces. Suppose that completely bounded maps $\sigma: A_1 \rightarrow A_2$ and $\tau: B_1 \rightarrow B_2$ satisfy $\sigma(S_1) \subseteq S_2$ and $\tau(T_1) \subseteq T_2$. Then*

$$(\sigma \otimes \tau)(F(S_1, T_1, A_1 \otimes B_1)) \subseteq F(S_2, T_2, A_2 \otimes B_2).$$

Proof. Let $x \in F(S_1, T_1, A_1 \otimes B_1)$. Let $\phi_2 \in A_2^*$ and let $\phi_1 := \phi_2 \circ \sigma \in A_1^*$. Then $(\phi_1 \otimes \text{id}_{B_1})(x) \in T_1$, thus

$$(\phi_2 \otimes \text{id}_{B_2})[(\sigma \otimes \tau)(x)] = \tau[(\phi_1 \otimes \text{id}_{B_1})(x)]$$

belongs to T_2 . Similarly for $\psi_2 \in B_2^*$, we have $(\text{id}_{A_2} \otimes \psi_2)[(\sigma \otimes \tau)(x)] \in S_2$. Thus $(\sigma \otimes \tau)(x) \in F(S_2, T_2, A_2 \otimes B_2)$. \square

Lemma 2.15. *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. Let $\sigma: A \rightarrow A$ and $\tau: B \rightarrow B$ be completely bounded maps. If σ restricts to the identity on S and τ on T , then $\sigma \otimes \tau$ restricts to the identity on $F(S, T, A \otimes B)$.*

Proof. For $x \in F(S, T, A \otimes B)$ and $\phi \in A^*$ and $\psi \in B^*$, we have

$$\begin{aligned} \langle \phi \otimes \psi, (\sigma \otimes \tau)(x) \rangle &= \langle \phi, \sigma[(\text{id}_A \otimes (\psi \circ \tau))(x)] \rangle \\ &= \langle \phi, (\text{id}_A \otimes (\psi \circ \tau))(x) \rangle \\ &= \langle \psi, \tau[(\phi \otimes \text{id}_B)(x)] \rangle \\ &= \langle \phi \otimes \psi, x \rangle. \end{aligned}$$

Thus $(\sigma \otimes \tau)(x) = x$. \square

Corollary 2.16 ((cf. [10, Lemma 2])). *For $i = 1, 2$, let $S \subseteq A_i$ and $T \subseteq B_i$ be operator spaces. Suppose that completely bounded maps $\sigma_1: A_1 \rightarrow A_2$, $\sigma_2: A_2 \rightarrow A_1$ and $\tau_1: B_1 \rightarrow B_2$, $\tau_2: B_2 \rightarrow B_1$ satisfy, for $i = 1, 2$, $\sigma_i(s) = s$ for $s \in S$ and $\tau_i(t) = t$ for $t \in T$, then $\sigma_1 \otimes \tau_1$ restricts to an isomorphism*

$$F(S, T, A_1 \otimes B_1) \cong F(S, T, A_2 \otimes B_2).$$

\square

Corollary 2.17 ((cf. [20, Proposition 3.7])). *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. If there exist completely bounded projections $A \rightarrow S$ and $B \rightarrow T$, then $(S, T, A \otimes B)$ has the slice map property:*

$$F(S, T, A \otimes B) = S \otimes T.$$

□

2.4. Intersections.

Lemma 2.18. *For operator spaces $S \subseteq A$ and $T \subseteq B$, we have*

$$F(S, T, A \otimes B) = F(A, T, A \otimes B) \cap F(S, B, A \otimes B).$$

More generally, for families of operator spaces $\{S_\alpha \subseteq A\}$ and $\{T_\beta \subseteq B\}$, we have

$$F(\cap_\alpha S_\alpha, \cap_\beta T_\beta, A \otimes B) = \cap_{\alpha, \beta} F(S_\alpha, T_\beta, A \otimes B).$$

Proof. Clear from the definitions. □

Corollary 2.19 ((cf. [23, Corollary 5])). *Let $S_1, S_2 \subseteq A$ and $T_1, T_2 \subseteq B$ be operator spaces. If $(S_1 \cap S_2, T_1 \cap T_2, A \otimes B)$ has the slice map property, then*

$$(S_1 \cap S_2) \otimes (T_1 \cap T_2) = (S_1 \otimes T_1) \cap (S_2 \otimes T_2).$$

Proof. We have

$$\begin{aligned} (S_1 \cap S_2) \otimes (T_1 \cap T_2) &\subseteq (S_1 \otimes T_1) \cap (S_2 \otimes T_2) \\ &\subseteq F(S_1, T_1, A \otimes B) \cap F(S_2, T_2, A \otimes B) \\ &= F(S_1 \cap S_2, T_1 \cap T_2, A \otimes B). \end{aligned}$$

□

2.5. Kernels.

Proposition 2.20. *Let A, B and D be operator spaces and let $\tau: B \rightarrow D$ be a completely bounded map. Then*

$$F(A, \ker(\tau), A \otimes B) = \ker(\text{id}_A \otimes \tau).$$

Proof. Let $x \in A \otimes B$. For $\phi \in A^*$, we have

$$\tau[(\phi \otimes \text{id}_B)(x)] = (\phi \otimes \text{id}_D)[(\text{id}_A \otimes \tau)(x)].$$

Thus $x \in F(A, \ker(\tau), A \otimes B)$ iff $x \in \ker(\text{id}_A \otimes \tau)$. □

Corollary 2.21 ((cf. [20, Corollary 1])). *Let A, B and D be operator spaces and let $\tau: B \rightarrow D$ be a completely bounded map. Then the equality*

$$A \otimes \ker(\tau) = \ker(\text{id}_A \otimes \tau)$$

holds if and only if the triple $(A, \ker(\tau), A \otimes B)$ has the slice map property. □

Theorem 2.22. *Let A, B, C and D be operator spaces and let $\{\sigma_\alpha: A \rightarrow C\}$ and $\{\tau_\beta: B \rightarrow D\}$ be families of completely bounded maps. Then*

$$F(\cap_\alpha \ker(\sigma_\alpha), \cap_\beta \ker(\tau_\beta), A \otimes B) = (\cap_\alpha \ker(\sigma_\alpha \otimes \text{id}_B)) \cap (\cap_\beta \ker(\text{id}_A \otimes \tau_\beta)).$$

Proof. Follows from Lemma 2.18 and Proposition 2.20. □

2.6. Relative Commutants. As a corollary we obtain the following.

Proposition 2.23. *Let $S, A \subseteq B(\mathcal{H})$ and $T, B \subseteq B(\mathcal{K})$ be operator spaces. Then we have*

$$F(S' \cap A, T' \cap B, A \otimes B) = (S \otimes \mathbb{C}1_{B(\mathcal{K})} + \mathbb{C}1_{B(\mathcal{H})} \otimes T)' \cap (A \otimes B).$$

Proof. For $s \in S$, let $\sigma_s := [-, s]: A \rightarrow B(\mathcal{H})$. Then σ_s is completely bounded and

$$\sigma_s \otimes \text{id}_B = [-, s \otimes 1_{B(\mathcal{K})}]: A \otimes B \rightarrow B(\mathcal{H}) \otimes B.$$

Moreover, we have

$$\bigcap_{s \in S} \ker(\sigma_s) = S' \cap A \quad \text{and}$$

$$\bigcap_{s \in S} \ker(\sigma_s \otimes \text{id}_B) = (S \otimes \mathbb{C}1_{B(\mathcal{K})})' \cap (A \otimes B).$$

Similary, $\tau_t := [-, t]: B \rightarrow B(\mathcal{K})$, $t \in T$, are completely bounded and

$$\bigcap_{t \in T} \ker(\tau_t) = T' \cap B \quad \text{and}$$

$$\bigcap_{t \in T} \ker(\text{id}_A \otimes \tau_t) = (\mathbb{C}1_{B(\mathcal{H})} \otimes T)' \cap (A \otimes B).$$

Thus Theorem 2.22 completes the proof. \square

Corollary 2.24 ((cf. [9, Theorem 1] and [20, Corollary 2])). *Let $S, A \subseteq B(\mathcal{H})$ and $T, B \subseteq B(\mathcal{K})$ and suppose $1_{B(\mathcal{H})} \in S$ and $1_{B(\mathcal{K})} \in T$. Then the equality*

$$(S' \cap A) \otimes (T' \cap B) = (S \otimes T)' \cap (A \otimes B)$$

holds if and only if the triple $(S' \cap A, T' \cap B, A \otimes B)$ has the slice map property.

Proof. Follows from Proposition 2.23, since under the unitality conditions, we have

$$(S \otimes T)' = (S \otimes \mathbb{C}1_{B(\mathcal{K})} + \mathbb{C}1_{B(\mathcal{H})} \otimes T)'.$$

\square

Corollary 2.25 ((cf. [9, Corollary 1])). *Let A and B be C^* -algebras. Then*

$$Z(A) \otimes Z(B) \cong Z(A \otimes B),$$

where Z denotes the center.

Proof. Follows from Corollary 2.24 and Remark 2.40. \square

2.7. Invariant Elements. The following can be deduced from Proposition 2.23. We give a direct proof.

Proposition 2.26. *Let A and B be operator spaces. Suppose that the group G acts on A and the group H acts on B by completely bounded maps. Then*

$$F(A^G, B^H, A \otimes B) = (A \otimes B)^{G \times H}.$$

Proof. For $g \in G$, let $\sigma_g := (\text{id}_A - g): A \rightarrow A$. Then σ_g is completely bounded and

$$\sigma_g \otimes \text{id}_B = (\text{id}_{A \otimes B} - 1_H \times g): A \otimes B \rightarrow A \otimes B.$$

Moreover, we have

$$\bigcap_{g \in G} \ker(\sigma_g) = A^G$$

$$\bigcap_{g \in G} \ker(\sigma_g \otimes \text{id}_B) = (A \otimes B)^{G \times \{1_H\}}.$$

Similarly, $\tau_h := (\text{id}_B - h): B \rightarrow B$, $h \in H$, are completely bounded and

$$\begin{aligned}\bigcap_{h \in H} \ker(\tau_h) &= B^H \\ \bigcap_{h \in H} \ker(\text{id}_A \otimes \tau_h) &= (A \otimes B)^{\{1_G\} \times H}.\end{aligned}$$

Now Theorem 2.22 completes the proof, since

$$(A \otimes B)^{G \times H} = (A \otimes B)^{G \times \{1_H\}} \cap (A \otimes B)^{\{1_G\} \times H}.$$

□

Corollary 2.27 ((cf. [23, Corollary 7])). *Let A and B be operator spaces. Suppose that group G acts on A and group H acts on B by completely bounded maps. Then the equality*

$$A^G \otimes B^H = (A \otimes B)^{G \times H}$$

holds if and only if the triple $(A^G, B^H, A \otimes B)$ has the slice map property.

2.8. Combinations.

Lemma 2.28. *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. Then*

$$F(S, T, A \otimes T) = (A \otimes T) \cap F(S, B, A \otimes B).$$

Proof. Clear since any element of T^* extends to an element of B^* . □

Lemma 2.29 ([12, Lemma 2.6]). *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. If $(A, T, A \otimes B)$ has the slice map property, then*

$$F(S, T, A \otimes T) = F(S, T, A \otimes B)$$

Proof. We have

$$\begin{aligned}F(S, T, A \otimes T) &= (A \otimes T) \cap F(S, B, A \otimes B) \\ &= F(A, T, A \otimes B) \cap F(S, B, A \otimes B) \\ &= F(S, T, A \otimes B),\end{aligned}$$

by Lemma 2.28 and Lemma 2.18. □

Proposition 2.30 ([12, Corollary 2.7]). *Let $S \subseteq A$ and $T \subseteq B$ be operator spaces. If $(A, T, A \otimes B)$ and $(S, T, A \otimes T)$ have the slice map property, then $(S, T, S \otimes B)$ and $(S, T, A \otimes B)$ also have the slice map property. Conversely, if $(S, T, A \otimes B)$ has the slice map property, then $(S, T, A \otimes T)$ and $(S, T, S \otimes B)$ also have the slice map property.*

Proof. Follows from Lemma 2.29 and the following commutative diagram of inclusions:

$$\begin{array}{ccc} F(S, T, A \otimes T) & \hookrightarrow & F(S, T, A \otimes B) \\ \uparrow & & \uparrow \\ S \otimes T & \hookrightarrow & F(S, T, S \otimes B) \end{array}$$

□

2.9. The Operator Approximation Property. In this subsection, we review the connection between the operator approximation property and the slice map property. This is partly for completeness and partly because we use it in Section 3. However, we have nothing new to add here to the excellent work of Kraus [15] and Haagerup-Kraus [6].

Definition 2.31. We say that A has the *slice map property* for B if $(A, T, A \otimes B)$ has the slice map property for all operator spaces $T \subseteq B$.

Definition 2.32. We write $F(A, B)$ for the space of finite-rank maps $A \rightarrow B$.

Lemma 2.33. *Finite-rank maps of operator spaces are completely bounded.* \square

Let A and B be operator spaces and let $x \in A \otimes B$. Define

$$\begin{aligned} F_B(x) &:= \{(\Phi \otimes \text{id}_B)(x) \mid \Phi \in F(A, A)\}^\perp \subseteq A \otimes B, \\ T_B(x) &:= \{(\phi \otimes \text{id}_B)(x) \mid \phi \in A^*\}^\perp \subseteq B. \end{aligned}$$

Then

$$F_B(x) = A \otimes T_B(x).$$

Moreover, we have

$$\begin{aligned} F(A, T, A \otimes B) &= \{x \in A \otimes B \mid T_B(x) \subseteq T\} \\ &= \{x \in A \otimes B \mid F_B(x) \subseteq A \otimes T\}. \end{aligned}$$

Lemma 2.34 ([15, Theorem 5.4]). *Let A and B be operator spaces. Then A has the slice map property for B if and only if $x \in F_B(x)$ for all $x \in A \otimes B$.*

Proof. (\Rightarrow): Let $x \in A \otimes B$. Clearly, $x \in F(A, T_B(x), A \otimes B)$. Since A has the slice map property for B , we have $F(A, T_B(x), A \otimes B) = A \otimes T_B(x) = F_B(x)$. Thus $x \in F_B(x)$.

(\Leftarrow): Let $T \subseteq B$ be operator subspace and let $x \in F(A, T, A \otimes B)$. Then $F_B(x) \subseteq A \otimes T$. Thus $x \in A \otimes T$. \square

Definition 2.35. An operator space B is *matrix stable* if for each $n \in \mathbb{N}$, there is a completely bounded surjection $B \rightarrow B \otimes M_n$.

Definition 2.36. We say that A has the *operator approximation property* (OAP) for B if there is a net $\Phi_\alpha \in F(A, A)$ such that $\Phi_\alpha \otimes \text{id}_B$ converges to $\text{id}_A \otimes \text{id}_B$ in point-norm topology.

Theorem 2.37 (cf. [15, Theorem 5.4]). *Let A and B be operator spaces. If A has the operator approximation property for B , then A has the slice map property for B . If B is matrix stable, then the converse also holds.*

Proof. The first statement is clear from Lemma 2.34.

Now we prove the second statement. Suppose B is stable. Let $x_1, \dots, x_n \in A \otimes B$ and let $\epsilon > 0$. Let $C := B \otimes M_n$ and choose a completely bounded surjection $\pi: B \rightarrow C$.

Let $x := x_1 \oplus \dots \oplus x_n \in A \otimes B \otimes M_n = A \otimes C$. Then there exists $y \in A \otimes B$ such that $x = (\text{id}_A \otimes \pi)(y)$. By Lemma 2.34, there is $\Phi \in F(A, A)$ such that $\|(\Phi \otimes \text{id}_B)(y) - y\| < \epsilon/(\|\pi\|_{\text{cb}} + 1)$. Then

$$\begin{aligned} \|(\Phi \otimes \text{id}_C)(x) - x\| &= \|(\Phi \otimes \text{id}_C)((\text{id}_A \otimes \pi)(y)) - (\text{id}_A \otimes \pi)(y)\| \\ &= \|(\text{id}_A \otimes \pi)((\Phi \otimes \text{id}_B)(y) - y)\| \\ &< \epsilon. \end{aligned}$$

It follows, for any $1 \leq k \leq n$, we have $\|(\Phi \otimes \text{id}_B)(x_k) - x_k\| < \epsilon$. \square

Definition 2.38. Let \mathcal{H} is a separable infinite dimensional Hilbert space. We say that A has the *operator approximation property* (OAP) if A has the OAP for $K(\mathcal{H})$ and the *strong operator approximation property* (SOAP) if A has the OAP for $B(\mathcal{H})$.

Lemma 2.39. *If an operator space has the strong OAP, then it has the OAP for any B .* \square

Remark 2.40. We have the implications

$$\text{CPAP} \implies \text{CBAP} \implies \text{strong OAP} \implies \text{OAP}.$$

Definition 2.41 ([6, Definition 1.1]). A countable discrete group G has the *approximation property* (AP) if the constant function 1 is in the $\sigma(M_0A(G), Q(G))$ -closure of $A(G)$ in $M_0A(G)$, where $A(G)$ is the Fourier algebra of G and $M_0A(G)$ is the space of completely bounded Fourier multipliers of G and $Q(G)$ is the standard predual of $M_0A(G)$.

Theorem 2.42 ([6, Theorem 2.1]). *A countable discrete group G has the AP if and only if its reduced group C^* -algebra $C_\lambda^*(G)$ has the (strong) OAP.*

Proof. See the original article [6] or [3, Section 12.4]. \square

3. THE INVARIANT TRANSLATION APPROXIMATION PROPERTY

In this section, we apply the results in Section 2 to the problem of studying the invariant part of the uniform Roe algebra.

3.1. Uniform Roe Algebras.

Definition 3.1. We say that a (countable discrete) metric space is of *bounded geometry* if for any $R > 0$, there is $N_R < \infty$ such that all balls of radius at most R have at most N_R elements.

Definition 3.2 ((cf. [17, Section 4.1] and [24, Section 2])). Let (X, d) be a metric space of bounded geometry and let $S \subseteq B(\mathcal{H})$ be a subset. For $R > 0$ and $M > 0$, let $A_{R,M}(X, S)$ denote the set of $X \times X$ -matrices with values in S satisfying

- (1) for any $x_1, x_2 \in X$ with $d(x_1, x_2) > R$, we have $a(x_1, x_2) = 0$ and
- (2) for any $x_1, x_2 \in X$, we have $\|a(x_1, x_2)\|_{B(\mathcal{H})} \leq M$.

Let $A(X, S) := \cup_{R,M} A_{R,M}(X, S)$. For $S = \mathbb{C}$, we write $A_{R,M}(X)$ and $A(X)$.

Lemma 3.3. *Under the natural action, elements of $A_{R,M}(X, S)$ act on $l^2(X, \mathcal{H}) \cong l^2X \otimes \mathcal{H}$ as bounded operators of norm at most $M \cdot N_R$.*

Proof. The action is given by

$$(a\xi)(x_1) := \sum_{x_2} a(x_1, x_2)\xi(x_2), \quad \xi \in l^2X \otimes \mathcal{H}.$$

Thus

$$\begin{aligned}
\|(a\xi)(x_1)\| &\leq \sum_{d(x_1, x_2) \leq R} \|a(x_1, x_2)\xi(x_2)\| \\
&\leq M \sum_{d(x_1, x_2) \leq R} \|\xi(x_2)\| \\
&\leq M \left(N_R \sum_{d(x_1, x_2) \leq R} \|\xi(x_2)\|^2 \right)^{1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|a\xi\|^2 &= \sum_{x_1} \|(a\xi)(x_1)\|^2 \\
&\leq M^2 N_R \sum_{x_1} \sum_{d(x_1, x_2) \leq R} \|\xi(x_2)\|^2 \\
&\leq M^2 N_R^2 \sum_{x_2} \|\xi(x_2)\|^2 \\
&= M^2 N_R^2 \|\xi\|^2.
\end{aligned}$$

□

Thus we identify $A(X, S) \subseteq B(l^2 X \otimes \mathcal{H})$.

Lemma 3.4. *For any $S \subseteq B(\mathcal{H})$, we have*

$$A(X) \times S \subseteq A(X, S) \subseteq A(X, \overline{S}) \subseteq \overline{A(X, S)}$$

in $B(l^2 X \otimes \mathcal{H})$.

Proof. Only the last inclusion needs checking. Take $a \in A_{R, M}(X, \overline{S})$. For each positive integer $n \geq 1$, we define $a_n \in A(X, S)$ as follows. For $x_1, x_2 \in X$, if $d(x_1, x_2) > R$, then let $a_n(x_1, x_2) = 0$, and if $d(x_1, x_2) \leq R$, then choose $a_n(x_1, x_2) \in S$ to satisfy $\|a_n(x_1, x_2) - a(x_1, x_2)\| \leq 1/n$. Then $a_n \in A_{R, M+1}(X, S)$ and $\|a_n - a\| \leq N_R/n$ by Lemma 3.3. Hence the sequence a_n converges to a in $B(l^2 X \otimes \mathcal{H})$ as $n \rightarrow \infty$ and thus $a \in \overline{A(X, S)}$. □

Definition 3.5 ((cf. [17, Section 4.4] [24])). Let X be a metric space of bounded geometry and let $S \subseteq B(H)$ be an operator space. The *uniform Roe operator space* $C_u^*(X, S)$ is the closure of $A(X, S)$ in $B(l^2 X \otimes \mathcal{H})$. For $S = \mathbb{C}$, we write $C_u^*(X)$.

Lemma 3.6. *For any operator space $S \subseteq B(\mathcal{H})$, we have*

$$C_u^*(X) \otimes S \subseteq C_u^*(X, S).$$

Proof. Follows from Lemma 3.4. □

3.2. Products. For metric spaces (X, d_X) and (Y, d_Y) , we equip the product $X \times Y$ with the metric

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

If X and Y are of bounded geometry, then so is $X \times Y$.

Lemma 3.7. *Let X and Y be metric spaces of bounded geometry and let $S \subseteq B(\mathcal{H})$ be a subset. Then for $R, R' > 0$ and $M, M' > 0$ we have a natural inclusion*

$$A_{R,M}(X) \times A_{R',M'}(Y, S) \subseteq A_{R'',M''}(X \times Y, S),$$

where $R'' := \max\{R, R'\}$ and $M'' := M \cdot M'$. In particular, we have

$$A(X) \times A(Y, S) \subseteq A(X \times Y, S)$$

in $B(l^2 X \otimes l^2 Y \otimes \mathcal{H})$.

Proof. Take $a \in A_{R,M}(X)$ and $b \in A_{R',M'}(Y, S)$ and let $R'' = \max\{R, R'\}$ and $M'' = M \cdot M'$. For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we define

$$(a \otimes b)((x_1, y_1), (x_2, y_2)) := a(x_1, x_2) \otimes b(y_1, y_2).$$

- (1) For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $d((x_1, y_1), (x_2, y_2)) > R'' = \max\{R, R'\}$, we have either $d(x_1, x_2) > R'' \geq R$ thus $a(x_1, x_2) = 0$, or $d(y_1, y_2) > R'' \geq R'$ thus $b(y_1, y_2) = 0$, hence $(a \otimes b)((x_1, y_1), (x_2, y_2)) = a(x_1, x_2) \otimes b(y_1, y_2) = 0$ and
- (2) For $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have $\|(a \otimes b)((x_1, y_1), (x_2, y_2))\| = \|a(x_1, x_2)\| \cdot \|b(y_1, y_2)\| \leq M \cdot M' = M''$.

Hence $a \otimes b$ belongs to $A_{R'',M''}(X \times Y, S)$. The last statement is clear. \square

Lemma 3.8. *Let X and Y be metric spaces of bounded geometry and let $S \subseteq B(\mathcal{H})$ be a subset. Then for $R > 0$ and $M > 0$, we have a natural inclusion*

$$A_{R,M}(X \times Y, S) \subseteq A_{R,M'}(X, A_{R,M}(Y, S)),$$

where $M' := M \cdot N_R^X$. In particular, we have

$$A(X \times Y, S) \subseteq A(X, A(Y, S))$$

in $B(l^2 X \otimes l^2 Y \otimes \mathcal{H})$.

Proof. Take $a \in A_{R,M}(X \times Y, S)$. Fix $x_1, x_2 \in X$ and consider $b_{x_1, x_2}(y_1, y_2) := a((x_1, y_1), (x_2, y_2))$. First we show that b_{x_1, x_2} is an element of $A_{R,M}(Y, S)$.

- (1) For $y_1, y_2 \in Y$ with $d(y_1, y_2) > R$, we have $d((x_1, y_1), (x_2, y_2)) \geq d(y_1, y_2) > R$, thus $b_{x_1, x_2}(y_1, y_2) = a((x_1, y_1), (x_2, y_2)) = 0$.
- (2) For $y_1, y_2 \in Y$, we have $\|b_{x_1, x_2}(y_1, y_2)\| = \|a((x_1, y_1), (x_2, y_2))\| \leq M$.

Now we show that b is an element of $A_{R,M'}(X, A_{R,M}(Y, S))$.

- (1) For $x_1, x_2 \in X$ with $d(x_1, x_2) > R$ and for any $y_1, y_2 \in Y$, we have $d((x_1, y_1), (x_2, y_2)) \geq d(x_1, x_2) > R$, thus

$$b_{x_1, x_2}(y_1, y_2) = a((x_1, y_1), (x_2, y_2)) = 0$$

i.e. $b_{x_1, x_2} = 0$

- (2) For $x_1, x_2 \in X$, we have $\|b_{x_1, x_2}\| \leq M \cdot N_R^X = M'$ by Lemma 3.3.

\square

Theorem 3.9. *Let X and Y be metric spaces of bounded geometry and let $S \subseteq B(H)$ be an operator space. Then we have a natural inclusion*

$$C_u^*(X) \otimes C_u^*(Y, S) \subseteq C_u^*(X \times Y, S) \subseteq C_u^*(X, C_u^*(Y, S)).$$

Proof. Follows from Lemmas 3.7 and 3.8 \square

3.3. The Invariant Translation Approximation Property. Let G be a countable discrete group. Let $|G|$ denote the metric space of bounded geometry on G associated to a proper length function. Then G acts on $|G|$ isometrically by right translations.

Example 3.10. Let G be a countable discrete group. Then $A(G)$ is the $*$ -algebra generated by $\mathbb{C}[G]$ and $l^\infty(G)$, and therefore $C_u^*|G| = C^*(l^\infty(G), \lambda(G))$. It is well-known that $C_u^*|G| \cong l^\infty(G) \rtimes_r G$ and $(C_u^*|G|)^G = L(G) \cap C_u^*|G|$, where $L(G)$ denote the von Neumann algebra generated by $C_\lambda^*(G)$. See [3, Section 5.1].

Definition 3.11. Let S be an operator space. We say G has the *invariant translation approximation property* (ITAP) for S , if we have

$$C_\lambda^*(G) \otimes S = C_u^*(|G|, S)^G.$$

Definition 3.12. We say G has the *(strong) invariant translation approximation property* if it has the ITAP for (all operator spaces $S \subseteq B(\mathcal{H})$) \mathbb{C} .

The following theorem connects the strong ITAP to the AP.

Theorem 3.13 ([24]). *Let G be a countable discrete group. Then G has the AP if and only if G is exact and has the strong ITAP.* \square

Proof. See [24]. \square

Theorem 3.14. *Any subgroup of a group with ITAP has ITAP.*

Proof. Let H be a subgroup of G and suppose that G has the ITAP. Using the coset decomposition of G one checks that we have inclusions $C_\lambda^*(H) \subseteq C_\lambda^*(G)$, $L(H) \subseteq L(G)$ and $C_u^*|H| \subseteq C_u^*|G|$. Now the multiplier with respect to the indicator function of H , being positive definite, induces conditional expectations $E_L: L(G) \rightarrow L(H)$, $E_u: C_u^*|G| \rightarrow C_u^*|H|$ and $E_\lambda: C_\lambda^*(G) \rightarrow C_\lambda^*(H)$. The restrictions of E_L and E_u to $C_\lambda^*(G)$ are equal to E_λ . Now let $x \in L(H) \cap C_u^*|H|$ then $x \in L(G) \cap C_u^*|G| = C_\lambda^*(G)$ and $E_L(x) = x$ as well as $E_u(x) = x$. It follows that $E_\lambda(x) = x$ i.e. $x \in C_\lambda^*(H)$ and thus H also has the ITAP. \square

Now we consider products.

Lemma 3.15. *Let G and H be countable discrete groups with proper length functions $l_G: G \rightarrow \mathbb{R}_{\geq 0}$ and $l_H: H \rightarrow \mathbb{R}_{\geq 0}$. Then $l_{G \times H}: G \times H \rightarrow \mathbb{R}_{\geq 0}$ given by $l_{G \times H}(g, h) = \max\{l_G(g), l_H(h)\}$ is a proper length function and $|G \times H| = |G| \times |H|$.*

Proof. First we check that l is a length function. Indeed, we have

$$l_{G \times H}(e, e) = \max\{l_G(e), l_H(e)\} = 0$$

and

$$\begin{aligned} l_{G \times H}(g^{-1}, h^{-1}) &= \max\{l_G(g^{-1}), l_H(h^{-1})\} \\ &= \max\{l_G(g), l_H(h)\} \\ &= l_{G \times H}(g, h) \end{aligned}$$

and finally

$$\begin{aligned}
l_{G \times H}(gg', hh') &= \max\{l_G(gg'), l_H(hh')\} \\
&\leq \max\{l_G(g) + l_G(g'), l_H(h) + l_H(h')\} \\
&\leq \max\{l_G(g), l_H(h)\} + \max\{l_G(g'), l_H(h')\} \\
&= l_{G \times H}(g, h) + l_{G \times H}(g', h').
\end{aligned}$$

Moreover, it is clear that l is proper. Finally, we have

$$\begin{aligned}
d_{|G \times H|}((g_1, h_1), (g_2, h_2)) &= l_{G \times H}(g_1 g_2^{-1}, h_1 h_2^{-1}) \\
&= \max\{l_G(g_1 g_2^{-1}), l_H(h_1 h_2^{-1})\} \\
&= \max\{d_{|G|}(g_1, g_2), d_{|H|}(h_1, h_2)\} \\
&= d_{|G| \times |H|}((g_1, h_1), (g_2, h_2)).
\end{aligned}$$

□

Proposition 3.16. *Let G and H be countable discrete groups. If $G \times H$ has the ITAP, then G and H also have the ITAP and the triple*

$$((C_u^*|G|)^G, (C_u^*|H|)^H, C_u^*|G| \otimes C_u^*|H|)$$

has the slice map property.

Proof. By Theorem 3.9, we have

$$\begin{aligned}
C_\lambda^*(G \times H) &= C_\lambda^*(G) \otimes C_\lambda^*(H) \\
&\subseteq (C_u^*|G|)^G \otimes (C_u^*|H|)^H \\
&\subseteq (C_u^*|G| \otimes C_u^*|H|)^{G \times H} \\
&\subseteq (C_u^*|G \times H|)^{G \times H} \\
&= C_\lambda^*(G \times H).
\end{aligned}$$

Thus G and H have the ITAP by Lemma 2.5 (this is also immediate from Theorem 3.14). The last statement follows from Corollary 2.27. □

Theorem 3.17. *Let G be a countable discrete group with the AP and let B be an operator space equipped with an action of a group H by completely bounded maps. Then we have*

$$C_\lambda^*(G) \otimes B^H = C_u^*(|G|, B)^{G \times H}.$$

Proof. Clearly, we have

$$C_\lambda^*(G) \otimes B^H \subseteq (C_u^*|G|)^G \otimes B^H \subseteq (C_u^*|G| \otimes B)^{G \times H} \subseteq C_u^*(|G|, B)^{G \times H}.$$

Since G has AP, it has the strong ITAP by Theorem 3.13, thus

$$C_u^*(|G|, B)^G = C_\lambda^*(G) \otimes B.$$

Moreover, the C^* -algebra $C_\lambda^*(G)$ has the strong OAP by Theorem 2.42, thus by Proposition 2.26, we have

$$(C_\lambda^*(G) \otimes B)^H = F(C_\lambda^*(G), B^H, C_\lambda^*(G) \otimes B) = C_\lambda^*(G) \otimes B^H.$$

It follows that

$$\begin{aligned} C_u^*(|G|, B)^{G \times H} &= (C_u^*(|G|, B)^G)^H \\ &\subseteq (C_\lambda^*(G) \otimes B)^H \\ &= C_\lambda^*(G) \otimes B^H. \end{aligned}$$

This completes the proof. \square

Theorem 3.18. *Let G and H be countable discrete groups. If G has the AP and H has the ITAP, then $G \times H$ has the ITAP.*

Proof. By Theorem 3.9, we have

$$C_u^*(G \times H) \subseteq C_u^*(|G|, C_u^*(|H|)).$$

Thus, by Theorem 3.17, we have

$$(C_u^*(G \times H))^{G \times H} \subseteq C_u^*(|G|, C_u^*(|H|))^{G \times H} = C_\lambda^*(G) \otimes (C_u^*(|H|))^H.$$

Since H has ITAP, we have $(C_u^*(|H|))^H = C_\lambda^*(H)$. Thus, we obtain

$$\begin{aligned} C_\lambda^*(G) \otimes C_\lambda^*(H) &= C_\lambda^*(G \times H) \\ &\subseteq (C_u^*(G \times H))^{G \times H} \\ &\subseteq C_\lambda^*(G) \otimes C_\lambda^*(H), \end{aligned}$$

so that $G \times H$ has the ITAP. \square

Corollary 3.19. *Let G and H be countable discrete groups. Suppose G has the AP. Then $G \times H$ has the ITAP if and only if H has the ITAP.* \square

4. THE FUBINI CROSSED PRODUCT

In this section, we study the crossed product version of the Fubini product.

4.1. The Fubini Crossed Product. Let G be a countable discrete group. A G -operator space is an operator space equipped with a completely isometric action of G .

Let $A \subseteq B(\mathcal{H})$ be an operator space equipped with a completely isometric action α of G . Define a new action π of A on $\mathcal{H} \otimes l^2 G$ by $\pi(a)(v \otimes \delta_g) := \alpha_{g^{-1}}(a)v \otimes \delta_g$ and let G act on $\mathcal{H} \otimes l^2 G$ by $\lambda(g)(v \otimes \delta_h) := v \otimes \delta_{gh}$.

The *reduced crossed product* $A \rtimes_r G$ is defined as the operator space spanned by $\{\pi(a)\lambda(g) \in B(\mathcal{H} \otimes l^2(G)) \mid a \in A, g \in G\}$.

Definition 4.1 ([14, Lemma 2.1]). For any $\psi \in B(l^2 G)_*$, there is a natural, completely bounded slice-map $\text{id}_A \rtimes_r \psi: A \rtimes_r G \rightarrow A$. If $S \subseteq A$ is a G -invariant operator subspace, then the *Fubini crossed product* $F(S, A \rtimes_r G)$ is defined as the set of all $x \in A \rtimes_r G$ such that $(\text{id}_A \rtimes_r \psi)(x) \in S$ for all $\psi \in B(l^2 G)_*$.

The slice map $\text{id}_A \rtimes_r \psi$ is given by the restriction of the von Neumann slice map which maps $A^{**} \overline{\otimes} B(l^2 G)$ to A^{**} .

Remark 4.2. We have

$$S \rtimes_r G \subseteq F(S, A \rtimes_r G) \subseteq A \rtimes_r G.$$

In fact, many of the formal properties of the Fubini product hold for Fubini crossed products, usually with the same proof.

Definition 4.3. We say that $(S, A \rtimes_r G)$ has the *slice map property* if

$$F(S, A \rtimes_r G) = S \rtimes_r G.$$

Lemma 4.4. *If the action of G on A is trivial, then*

$$F(S, A \rtimes_r G) = F(S, C_\lambda^*(G), A \otimes C_\lambda^*(G)).$$

Proof. Since the action of G on A is trivial, we have $A \rtimes_r G = A \otimes C_\lambda^*(G)$. The inclusion $C_\lambda^*(G) \hookrightarrow B(l^2 G)$ gives the diagram

$$\begin{array}{ccc} B(l^2 G)^* & \twoheadrightarrow & C_\lambda^*(G)^* \\ \uparrow & \nearrow & \\ B(l^2 G)_* & & \end{array}$$

Hence we have $F(S, C_\lambda^*(G), A \otimes C_\lambda^*(G)) \subseteq F(S, A \rtimes_r G)$. By Goldstine's theorem for any functional $\psi \in C_\lambda^*(G)^*$, there exists a bounded net $\psi_n \in B(l^2(G))_*$ with $\|\psi_n\| \leq \|\psi\|$, converging to ψ in the weak* topology. Now it is easy to see that for any $x \in A \otimes C_\lambda^*(G)$, the elements $\text{id}_A \otimes \psi_n(x)$ converge to $\text{id}_A \rtimes \psi(x)$ in norm. It follows that $F(S, A \rtimes_r G) \subseteq F(S, C_\lambda^*(G), A \otimes C_\lambda^*(G))$. \square

Example 4.5. Let $l^\infty(G)$ act on $l^2 G$ by multiplication. Then G acts on $l^\infty(G)$ by left multiplication and $c_0(G) \subseteq l^\infty(G)$ is an invariant C^* -subalgebra. The Fubini crossed product $F(c_0(G), l^\infty(G) \rtimes_r G)$ is the ideal of all ghost operators on $|G|$. Thus G is exact if and only if $(c_0(G), l^\infty(G) \rtimes_r G)$ has the slice map property by [18].

Lemma 4.6. *For families of G -invariant operator subspaces $\{S_\alpha \subseteq A\}$ we have*

$$F(\cap_\alpha S_\alpha, A \rtimes_r G) = \cap_\alpha F(S_\alpha, A \rtimes_r G).$$

\square

Proposition 4.7 ((cf. [14, Proposition 2.2])). *Let B and D be G -operator spaces and let $\sigma: B \rightarrow D$ be a completely bounded G -equivariant map. Then*

$$F(\ker(\sigma), B \rtimes_r G) = \ker(\sigma \rtimes_r G).$$

\square

Theorem 4.8. *Let G and H be countable discrete groups and let A be a $(G \times H)$ -operator space. Then*

$$F(A^H, A \rtimes_r G) = (A \rtimes_r G)^H.$$

Proof. For $h \in H$, let $\alpha_h: A \rightarrow A$ denote the action by h . Then α_h is G -equivariant. Let $\sigma_h := \text{id}_A - \alpha_h$. Then $\sigma_h \rtimes_r G = \text{id}_{A \rtimes_r G} - \alpha_h \rtimes_r G: A \rtimes_r G \rightarrow A \rtimes_r G$. The proof follows from Proposition 4.7 and Lemma 4.6, since

$$\begin{aligned} \cap_{h \in H} \ker(\sigma_h) &= A^H, \\ \cap_{h \in H} \ker(\sigma_h \rtimes_r G) &= (A \rtimes_r G)^H. \end{aligned}$$

\square

Example 4.9. Let $l^\infty(G)$ act on l^2G by multiplication. Then G acts on $l^\infty(G)$ by left multiplication. As already pointed out $C_u^*[G] \cong l^\infty(G) \rtimes_r G$; moreover, thinking of \mathbb{C} as embedded into $l^\infty(G)$ via the constant functions we have $C_u^*(|G|)^G = F(\mathbb{C}, l^\infty(G) \rtimes_r G)$. Thus G has the ITAP if and only if $(\mathbb{C}, l^\infty(G) \rtimes_r G)$ has the slice map property.

Proposition 4.10. *Let G be a discrete group with the AP. Then for any $S \subseteq A$, we have $F(S, A \rtimes_r G) = S \rtimes_r G$.*

Proof. Since G has the AP there exists a net (u_α) in $M_0A(G) \cap c_c(G)$ converging to $1 \in M_0A(G)$ in the $\sigma(M_0A(G), Q(G))$ topology. As explained in [24], this implies that the net of Schur multipliers $\hat{M}_{u_\alpha} \in CB(C_u^*(|G|, A))$ given by $\hat{M}_{u_\alpha}([a_{s,t}]) = [u_\alpha(st^{-1})a_{s,t}]$ converges to the identity map in the point norm topology. Fixing a faithful representation $A \hookrightarrow B(K)$ we can think of $A \rtimes_r G \subseteq B(K \otimes \ell^2(G))$ as matrices indexed by G with entries in A . Since the $\langle \delta_s, \cdot \delta_t \rangle$ is a normal functional on $B(\ell^2(G))$ whose slice map gives the s, t entry of the matrix in $B(K \otimes \ell^2(G))$ we can characterise $F(S, A \rtimes_r G)$ as those matrices in $A \rtimes_r G$ with entries in S . Thus it is clear that $F(S, A \rtimes_r G) \subseteq C_u^*(|G|, A)$. Moreover, since each u_α has finite support, it is easy to check that $\hat{M}_{u_\alpha}(F(S, A \rtimes_r G))$ is contained in $\text{span}\{\pi(s)\lambda(g) \mid s \in S, g \in G\} \subseteq S \rtimes_r G$. It follows that $S \rtimes_r G \ni \hat{M}_\alpha(x) \rightarrow x$ for any $x \in F(S, A \rtimes_r G)$, concluding the proof. \square

Proposition 4.11. *If $F(S, A \rtimes_r G) = S \rtimes_r G$ for all $S \subseteq A$, then G has the AP.*

Proof. Considering trivial actions we see that $C_\lambda^*(G)$ has the slice map property for any operator space A by Lemma 4.4, thus have the OAP by Kraus' theorem (Theorem 2.37). Hence G has the AP by Haagerup-Kraus theorem (Theorem 2.42). \square

4.2. Functoriality.

Lemma 4.12. *Let $S \subseteq A$ and $T \subseteq B$ be G -operator spaces. Suppose that a completely bounded map $\sigma: A \rightarrow B$ maps $\sigma(S) \subseteq T$. Then*

$$(\sigma \rtimes_r G)(F(S, A \rtimes_r G)) \subseteq F(T, B \rtimes_r G).$$

Proof. Let $x \in F(S, A \rtimes_r G)$. For any $\psi \in B(l^2G)_*$, we have $(\text{id}_A \rtimes_r \psi)(x) \in S$, thus

$$(\text{id}_A \rtimes_r \psi)[(\sigma \rtimes_r G)(x)] = \sigma[(\text{id}_A \rtimes_r \psi)(x)]$$

belongs to T . Hence the lemma holds. \square

Lemma 4.13. *Let $S \subseteq A$ be G -operator spaces. Let $\sigma: A \rightarrow A$ be a completely bounded G -map. If σ restricts to the identity on S , then $\sigma \rtimes_r G$ restricts to the identity on $F(S, A \rtimes_r G)$.*

Proof. For $x \in F(S, A \rtimes_r G)$ and $\psi \in B(l^2G)_*$, we have

$$(\text{id}_A \rtimes_r \psi)[(\sigma \rtimes_r G)(x)] = \sigma[(\text{id}_A \rtimes_r \psi)(x)] = (\text{id}_A \rtimes_r \psi)(x).$$

Thus $(\sigma \rtimes_r G)(x) = x$. \square

Corollary 4.14 ((cf. [10, Lemma 2])). *Let $S \subseteq A$ and $S \subseteq B$ be G -operator spaces. If $\phi: A \rightarrow B$ and $\psi: B \rightarrow A$ are completely bounded G -maps such that $(\psi \circ \phi)|_S = \text{id}_S$ and $(\phi \circ \psi)|_S = \text{id}_S$, then there is a completely bounded G -isomorphism $F(S, A \rtimes_r G) \rightarrow F(S, B \rtimes_r G)$ which is the identity on $S \rtimes_r G$.* \square

Injective envelopes of C^* -algebras, operator systems and operator spaces have been considered by various authors [7, 8, 2, 5]. The G -injective envelope has only been defined for operator systems in the literature. However, the definitions and constructions are all analogous. First one has to find an injective extension of the given object in the appropriate category and then minimise it in such a way that uniqueness is automatic. Therefore in the following, we omit the proofs. For a G -operator space S , we denote the G -injective envelope by $I_G(S)$. We write $I(S)$ if G is trivial.

Lemma 4.15 ((cf. [10, Theorem 4])). *Let S be a G -operator system and let A be a G -injective operator system containing S . Then the Fubini crossed product $F(S, A \rtimes_r G)$ is independent of A .* \square

Definition 4.16. The Fubini crossed product $F(S, A \rtimes_r G)$ is called the *universal Fubini crossed product* of S by G and denoted $F(S, G)$. It is the largest Fubini crossed product of S .

We say that S has the *universal slice map property* for G if $S \rtimes_r G = F(S, G)$.

Example 4.17. For a discrete group G , the space $c_0(G)$ has the universal slice map property if and only if G is exact and the space \mathbb{C} has the universal slice map property if and only if G has the ITAP.

Theorem 4.18. *Let G be a discrete group. Then $(C_u^*[G])^G \subseteq I(C_\lambda^*(G))$.* \square

Proof. We only sketch the proof.

Since $l^\infty(G)$ is G -injective, we see that $F(\mathbb{C}, G) \cong F(\mathbb{C}, l^\infty(G) \rtimes_r G) \cong (C_u^*[G])^G$ by Lemma 4.15 and Example 4.9. On the other hand, for any S we have $S \rtimes_r G \subseteq F(S, G) \subseteq I_G(S) \rtimes_r G \subseteq I(S \rtimes_r G)$. \square

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